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A Note on Optimal Foldover Four-level Factorials

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Abstract The foldover is a quick and useful technique in construction of fractional factorial designs, which typically releases aliased factors or interactions. The issue of employing the uniformity criterion measured by the centered L_2-discrepancy to assess the optimal foldover plans was studied for four-level design. A new analytical expression and a new lower bound of the centered L_2-discrepancy for four-level combined design under a general foldover plan are respectively obtained. A necessary condition for the existence of an optimal foldover plan meeting this lower bound was described. An algorithm for searching the optimal four-level foldover plans is also developed. Illustrative examples are provided, where numerical studies lend further support to our theoretical results. These results may help to provide some powerful and efficient algorithms for searching the optimal four-level foldover plans.

Keywords Centered L_2-discrepancy, foldover plan, optimal foldover plan, combined design, lower bounds

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1 Introduction

The foldover is a quick and useful technique in construction of fractional factorial designs, which typically releases aliased factors or interactions. The uniformity criterion has firstly been employed to assess the optimal foldover plan by [9]. [9] firstly used the uniformity criterion measured by centered L_2-discrepancy to search the optimal foldover plan (see Definition 2.1). [10, 11] obtained some lower bounds of centered L_2-discrepancy of combined designs when all Hamming distances between any distinct pair of runs of the two-level initial design are equal. Subsequently, [12] obtained some lower bounds of various discrepancies of combined designs for general case. [14] further extended the ones of [12] for symmetric designs to a set of asymmetric designs under a special case, which the last p-level ($p \geq 2$) of the foldover design is retained as the same in the original design. [13] obtained lower bound of wrap-around L_2-discrepancy for mixed two and three-level combined design. Recently, [3] studied the issue of the optimal foldover plans for three-level designs in view of the uniformity criterion measured by the Lee discrepancy. Finally, [5, 7] obtained a new look for optimal foldover two-level and three-level designs in terms of uniformity criteria measured by the commonly used discrepancies criteria in the literature with a comparison study, respectively.

There are many measures to assess the uniformity of various designs. In this work, we consider a widely used measures of uniformity, the centered L_2-discrepancy. It is an important issue to find good lower bounds for the discrepancy measure of uniformity, because lower bounds can be used as benchmarks not only in searching for optimal designs but also in helping to validate that some good designs are in fact optimal. A design whose discrepancy value achieves a sharp lower bound is an optimal design with respect to this discrepancy. There has been considerable interests in trying to explore lower bounds for various discrepancies. Many authors tried to find good lower bounds for various discrepancies. Recently, [2, 4] obtained a new lower bound for four-level and also the first lower bound for mixed two and three-level of the centered L_2-discrepancy based on U-type design respectively. Finally [6] obtained new lower bounds of the mixture discrepancy for two-, three- and four-level U-type designs.

In this article, we extend the work of [2] for U-type design to combined design. This paper aims to study the issue of optimal foldover plans for four-level designs, assessed by the uniformity criterion in terms of the centered L_2-discrepancy. Actually the foldover plan considered in this paper is 4^{-1} fraction of the foldover design in which the number of level combinations of the combined design is more economic, and the results are suitable to regular or nonregular designs.

The next section will describe some notations which are useful throughout this paper. In Section 3, a new formulation of the centered L_2-discrepancy for four-level combined design is obtained. In Section 4, a new lower bound for the centered L_2-discrepancy for four-level combined design is provided. An algorithm for searching the optimal four-level foldover plans is also developed in Section 5. Illustrative examples are provided in Section 6, where numerical studies lend further support to our theoretical results. We close through the conclusion and discussion in Section 7.

2 Notations and Preliminaries

Consider a class of n runs and m factors with four-level U-type designs, denoted as $U(n; 4^m)$. A design $d \in U(n; 4^m)$ can be presented as an $n \times m$ matrix with entries $0, 1, 2, 3$, with each element occurring equally often in each column. Let $d \in U(n;4^m)$ and let d be regarded as a set of m columns $d=(x^1,x^2,\ldots,x^m)$, where $x^j=(x_{1j},\ldots,x_{nj})'$ is the j-th column of d, $j=1,\ldots,m$. Define $\Gamma=\{\gamma=(\gamma_1,\ldots,\gamma_m)\,|\,\gamma_j=0,1,2,3;1\le j\le m\}$. Then for any $\gamma=(\gamma_1,\ldots,\gamma_m)\in\Gamma$, it defines a foldover plan for design $d\in U(n;4^m)$, for $1\le j\le m$, the j-th factor is mapped to $(x_{1j}+\gamma_j,\ldots,x_{nj}+\gamma_j)'$ (mod 4).

For any design $d\in U(n;4^m)$ and a foldover plan $\gamma\in\Gamma$, the foldover design, denoted by d_γ, is obtained by mapping the columns in d defined above according to the foldover plan γ. Thus, it is to be noted that each foldover design is generated by a foldover plan. The full design obtained by augmenting the runs of the foldover design d_γ to those of the original design d is called as a combined design, denoted by $d(\gamma)$, that is, $d(\gamma)=(d'\ d'_\gamma)'$.

In this paper, we use the centered L_2-discrepancy to evaluate the optimal foldover plan. For a design $d\in U(n;4^m)$, the centered L_2-discrepancy measure of uniformity, denoted as $\mathrm{CD}_2(d)$, can be expressed in the following closed form:

$$[\mathrm{CD}_2(d)]^2=\left(\frac{13}{12}\right)^m-\frac{2}{n}\sum_{i=1}^n\prod_{k=1}^m\Xi_{ik}+\frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n\prod_{k=1}^m\Xi_{ijk},$$

where

$$\Xi_{ik}=1+\frac{1}{2}\left|u_{ik}-\frac{1}{2}\right|-\frac{1}{2}\left|u_{ik}-\frac{1}{2}\right|^2,\ \Xi_{ijk}=1+\frac{1}{2}\left|u_{ik}-\frac{1}{2}\right|+\frac{1}{2}\left|u_{jk}-\frac{1}{2}\right|-\frac{1}{2}|u_{ik}-u_{jk}|$$

and $u_{ik}=(2x_{ik}+1)/8, x_{ik}\in\{0,1,2,3\}, i=1,\ldots,n, k=1,\ldots,m$.

Under a foldover plan γ, the squared centered L_2-discrepancy of the combined design $d(\gamma)$, denoted by $[\mathrm{CD}_2(d(\gamma))]^2$, can be calculated by the following formula:

$$[\mathrm{CD}_2(d(\gamma))]^2=\left(\frac{13}{12}\right)^m-\frac{2}{n}\sum_{i=1}^n\prod_{k=1}^m\Xi_{ik}+\frac{1}{2n^2}\sum_{i=1}^n\sum_{j=1}^n\left[\prod_{k=1}^m\Xi_{ijk}+\prod_{k=1}^m\Xi_{ijk}(\gamma_k)\right],\quad(2.1)$$

where $\Xi_{ijk}(\gamma_k)=1+\frac{1}{2}|u_{ik}-\frac{1}{2}|+\frac{1}{2}|u_{jk}-\frac{1}{2}|-\frac{1}{2}|u_{ik}-u_{jk}^{(\gamma_k)}|$ and $u_{jk}^{(\gamma_k)}=\frac{2x_j^k(\gamma_k)+1}{8}, j=1,\ldots,n, k=1,\ldots,m, x_j^k(\gamma_k)=(x_{jk}+\gamma_k)$ (mod 4).

Let Γ_t be the set of all elements $\gamma\in\Gamma$ such that γ exactly has t nonzero components among $\{\gamma_1,\ldots,\gamma_m\}$. For $\gamma=(\gamma_1,\gamma_2,\ldots,\gamma_m)\in\Gamma_t$, let $E_t=\{v|\gamma_v\ne0, v=1,\ldots,m\}$ of cardinality t, and $\bar{E}_t=\{1,2,\ldots,m\}-E_t$. For any $0\le t\le m$ and foldover plan $\gamma\in\Gamma_t$, we can rewrite (2.1) as follows.

$$\begin{aligned}[\mathrm{CD}_2(d(\gamma))]^2=&\left(\frac{13}{12}\right)^m-\frac{2}{n}\sum_{i=1}^n\prod_{k=1}^m\Xi_{ik}\\&+\frac{1}{2n^2}\sum_{i=1}^n\sum_{j=1}^n\left[\prod_{k=1}^m\Xi_{ijk}+\prod_{k\in E_t}\Xi_{ijk}(\gamma_k)\prod_{k\in\bar{E}_t}\Xi_{ijk}\right].\end{aligned}\quad(2.2)$$

Definition 2.1 *The optimal foldover plan in this paper is defined as the foldover plan such that its combined design achieves the lower bound of the centered L_2-discrepancy for combined designs, i.e., for any design $d\in U(n;4^m)$, the optimal foldover plan γ^* is defined as the foldover plan γ^*, such that*

$$[\mathrm{CD}_2(d(\gamma^*))]^2=\min\left\{[\mathrm{CD}_2(d(\gamma))]^2:\gamma\in\bigcup_{t=0}^m\Gamma_t\right\}.$$

3 Auxiliary Results

In this section, a new analytical expression of the squared centered L_2-discrepancy for four-level combined designs is obtained. We present a new formulation for $[\mathrm{CD}_2(d(\gamma))]^2$, based on the row distance of design d.

For any $0 \le t \le m$ and foldover plan $\gamma \in \Gamma_t$, let A and $\bar{A}$ respectively be the subdesigns with the column groups E_t and $\bar{E}_t$ in the original design d and let A_γ be the subdesign with the column groups E_t in the foldover design d_γ corresponding to γ. Let R_{ij} be the number of places where the entries of the i-th and the j-th rows of the design d coincide and $x_{ik} = x_{jk} \in \{1,2\}$, $R_{ij}^{\bar{\gamma}}$ be the number of places where the entries of the i-th and the j-th rows of the subdesign $\bar{A}$ coincide and $x_{ik} = x_{jk} \in \{1,2\}$ and let R_{ij}^{γ} be the number of places where the entries of the i-th row of the subdesign A and the j-th row of the subdesign A_γ coincide and $x_{ik} = x_{jk} \in \{1,2\}$. Similarly, let r_{ij}, r_{ij}^{γ} and $r_{ij}^{\bar{\gamma}}$, respectively have the same definitions as R_{ij}, R_{ij}^{γ} and $R_{ij}^{\bar{\gamma}}$ for $x_{ik} = x_{jk} \in \{0,3\}$. Let ζ_{ij} be the number of places where the entries of the i-th and the j-th rows of the design d different and $(x_{ik}, x_{jk}) \in \{(0,1),(2,3),(1,0),(3,2)\}$, $\zeta_{ij}^{\bar{\gamma}}$ be the number of places where the entries of the i-th and the j-th rows of the subdesign $\bar{A}$ different and $(x_{ik}, x_{jk}) \in \{(0,1),(2,3),(1,0),(3,2)\}$ and let ζ_{ij}^{γ} be the number of places where the entries of the i-th row of the subdesign A and the j-th row of the subdesign A_γ different and $(x_{ik}, x_{jk}) \in \{(0,1),(2,3),(1,0),(3,2)\}$.

Then, it is easy to observe the following lemma, which is a useful improvement of Lemma 1 in [2], we extend their results from U-type design to combined design.

Lemma 3.1 *For any four-level design $d \in U(n;4^m)$ and any foldover plan $\gamma \in \Gamma_t$, we have*

(1) $\sum_{i=1}^n \sum_{j\neq i}^n r_{ij} = \sum_{i=1}^n \sum_{j\neq i}^n R_{ij} = \frac{mn(n-4)}{8}, \sum_{i=1}^n r_{ii} = \sum_{i=1}^n R_{ii} = \frac{mn}{2},\ R_{ii} + r_{ii} = m;$

(2) $\sum_{i=1}^n \sum_{j\neq i}^n r_{ij}^{\bar{\gamma}} = \sum_{i=1}^n \sum_{j\neq i}^n R_{ij}^{\bar{\gamma}} = \frac{n(m-t)(n-4)}{8}, \sum_{i=1}^n r_{ii}^{\bar{\gamma}} = \sum_{i=1}^n R_{ii}^{\bar{\gamma}} = \frac{n(m-t)}{2};$

(3) $\sum_{i=1}^n \sum_{j\neq i}^n r_{ij}^{\gamma} = \sum_{i=1}^n \sum_{j\neq i}^n R_{ij}^{\gamma} = \frac{n^2 t}{8}, r_{ii}^{\gamma} = R_{ii}^{\gamma} = 0;$

(4) $\sum_{i=1}^n \sum_{j\neq i}^n \zeta_{ij} = \frac{mn^2}{4}, \zeta_{ii} = 0;$

(5) $\sum_{i=1}^n \sum_{j\neq i}^n \zeta_{ij}^{\bar{\gamma}} = \frac{n^2(m-t)}{4}, \zeta_{ii}^{\bar{\gamma}} = 0;$

(6) $\sum_{i=1}^n \sum_{j\neq i}^n \zeta_{ij}^{\gamma} = \frac{n^2 t - 2\rho n}{4}, \sum_{i=1}^n \zeta_{ii}^{\gamma} = \frac{\rho n}{2}, \rho = \sharp\{\gamma_i : \gamma_i = 1,3\}.$

On the other hand, we can give another expressions of $[\mathrm{CD}_2(d(\gamma))]^2$ based on Lemma 3.1.

Theorem 3.2 *For any four-level design $d \in U(n;4^m)$ and any foldover plan $\gamma \in \Gamma_t$, then*

$$\begin{aligned}
[\mathrm{CD}_2(d(\gamma))]^2 = {} & \left(\frac{13}{12}\right)^m - \frac{2}{n}\sum_{i=1}^n \left(\frac{143}{128}\right)^{r_{ii}} \left(\frac{135}{128}\right)^{R_{ii}} + \frac{1}{2n^2}\sum_{i=1}^n\sum_{j=1}^n \left(\frac{11}{8}\right)^{r_{ij}} \left(\frac{9}{8}\right)^{\xi_{ij}} \\
& + \frac{1}{2n^2}\sum_{i=1}^n\sum_{j=1}^n \left(\frac{11}{8}\right)^{r_{ij}^{\gamma} + r_{ij}^{\bar{\gamma}}} \left(\frac{9}{8}\right)^{\xi_{ij}^{\gamma} + \xi_{ij}^{\bar{\gamma}}},
\end{aligned} \tag{3.1}$$

where $\xi_{ij}^z = R_{ij}^z + \zeta_{ij}^z$.

Proof Following (2.2), we take u_{ik} and $u_{jk}^{(\gamma_k)} \in \left\{\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}\right\}$, $k = 1, \ldots, m$. Note that

$$\Xi_{ik} = \begin{cases} \dfrac{143}{128}, & x_{ik} \in \{0,3\}; \\ \dfrac{135}{128}, & x_{ik} \in \{1,2\}; \end{cases}$$

$$\Xi_{ijk}=\begin{cases}\dfrac{11}{8}, & x_{ik}=x_{jk}\in\{0,3\};\\ \dfrac{9}{8}, & x_{ik}=x_{jk}\in\{1,2\},\ (x_{ik},x_{jk})\in\{(0,1),(2,3),(1,0),(3,2)\};\\ 1, & \text{otherwise};\end{cases}$$

$$\Xi_{ijk}(\gamma_k)=\begin{cases}\dfrac{11}{8}, & x_{ik}=x_j^k(\gamma_k)\in\{0,3\};\\ \dfrac{9}{8}, & x_{ik}=x_j^k(\gamma_k)\in\{1,2\},\ (x_{ik},x_{jk})\in\{(0,1),(2,3),(1,0),(3,2)\};\\ 1, & \text{otherwise}.\end{cases}$$

Then, we have

$$\prod_{k=1}^{m}\Xi_{ik}=\left(\frac{143}{128}\right)^{r_{ii}}\left(\frac{135}{128}\right)^{R_{ii}},\quad \prod_{k=1}^{m}\Xi_{ijk}=\left(\frac{11}{8}\right)^{r_{ij}}\left(\frac{9}{8}\right)^{R_{ij}+\zeta_{ij}} \tag{i}$$

and

$$\prod_{k\in\bar{E}_t}\Xi_{ijk}=\left(\frac{11}{8}\right)^{r_{ij}^{\bar{\gamma}}}\left(\frac{9}{8}\right)^{R_{ij}^{\bar{\gamma}}+\zeta_{ij}^{\bar{\gamma}}},\quad \prod_{k\in E_t}\Xi_{ijk}(\gamma_k)=\left(\frac{11}{8}\right)^{r_{ij}^{\gamma}}\left(\frac{9}{8}\right)^{R_{ij}^{\gamma}+\zeta_{ij}^{\gamma}}. \tag{ii}$$

Substituting (i) and (ii) into (2.2), then (3.1) holds, which completes the proof. □

Remark 3.3 From Lemma 3.1, (3.1) can be expressed in the following useful form

$$\begin{aligned}[\mathrm{CD}_2(d(\gamma))]^2=&\left(\frac{13}{12}\right)^{m}-\frac{2}{n}\left(\frac{135}{128}\right)^{m}\sum_{i=1}^{n}\left(\frac{143}{135}\right)^{r_{ii}}+\frac{1}{2n^2}\left(\frac{9}{8}\right)^{m}\sum_{i=1}^{n}\left(\frac{11}{9}\right)^{r_{ii}}\\ &+\frac{1}{2n^2}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left(\frac{11}{8}\right)^{r_{ij}}\left(\frac{9}{8}\right)^{\xi_{ij}}+\frac{1}{2n^2}\sum_{i=1}^{n}\left(\frac{11}{8}\right)^{r_{ii}^{\bar{\gamma}}}\left(\frac{9}{8}\right)^{R_{ii}^{\bar{\gamma}}+\zeta_{ii}^{\gamma}}\\ &+\frac{1}{2n^2}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\left(\frac{11}{8}\right)^{r_{ij}^{\gamma}+r_{ij}^{\bar{\gamma}}}\left(\frac{9}{8}\right)^{\xi_{ij}^{\gamma}+\xi_{ij}^{\bar{\gamma}}}.\end{aligned} \tag{3.2}$$

From (3.2), we will provide the lower bound of $[\mathrm{CD}_2(d(\gamma))]^2$ which depends on the fact in the following lemmas.

Lemma 3.4 ([1]) *Suppose* $\sum_{i=1}^{n}y_i=r$ *and* y_i*'s are nonnegative, then for any positive integer* θ*, we have*

$$\sum_{i=1}^{n}y_i^{\theta}\geq pw^{\theta}+q(w+1)^{\theta},$$

where p *and* q *are integers such that* $p+q=n$, $w=\lfloor\frac{r}{n}\rfloor$, $pw+q(w+1)=r$ *and* $\lfloor\phi\rfloor$ *means the largest integer contained in* ϕ.

Finally, [2] have done some useful improvements on Lemma 3.4 as the following lemma.

Lemma 3.5 ([2]) *Suppose* $z_l=ax_l+by_l$, x_l, y_l*'s are nonnegative,* a, b *are positive constants,* $\sum_{l=1}^{n}x_l=r_1$, $\sum_{l=1}^{n}y_l=r_2$, p_s *and* q_s, $s=1,2$ *are integers such that* $p_s+q_s=n$, $h_s=\lfloor\frac{r_s}{n}\rfloor$ *and* $p_sh_s+q_s(h_s+1)=r_s$, $s=1,2$, *then for any positive integer* θ*, we have*

$$\sum_{l=1}^{n}z_l^{\theta}\geq\begin{cases}q_1w_1^{\theta}+q_2w_2^{\theta}+(p_1-q_2)w_3^{\theta}, & p_1>q_2,\\ p_2w_1^{\theta}+p_1w_2^{\theta}+(q_2-p_1)w_4^{\theta}, & p_1\leq q_2,\end{cases}$$

where $w_1 = a(h_1+1)+bh_2,\ w_2 = ah_1+b(h_2+1),\ w_3 = ah_1+bh_2,\ w_4 = a(h_1+1)+b(h_2+1)$ *and* $q_1w_1+q_2w_2+(p_1-q_2)w_3 = p_2w_1+p_1w_2+(q_2-p_1)w_4 = \sum_{l=1}^n z_l = ar_1+br_2$.

4 Main Results

Now, before we get our main results of this paper, we must give a further analysis of the above expression (3.2). Utilizing the same technique as in [2] we define a function

$$f(x) = \frac{1}{9n^2}\left(\frac{9}{8}\right)^m \left(\frac{11}{9}\right)^x - \frac{16}{135n}\left(\frac{135}{128}\right)^m \left(\frac{143}{135}\right)^x.$$

Obviously, the function $f(x)$ has a single root, i.e.,

$$x_0 = \frac{\log\left(\frac{16n}{15}\right) + m\log\left(\frac{16}{15}\right)}{\log\left(\frac{15}{13}\right)}$$

and the first derivative of $f(x)$,

$$\frac{df(x)}{dx} = \frac{1}{9n^2}\left(\frac{9}{8}\right)^m \left(\frac{11}{9}\right)^x \log\left(\frac{11}{9}\right) - \frac{16}{135n}\left(\frac{135}{128}\right)^m \left(\frac{143}{135}\right)^x \log\left(\frac{143}{135}\right)$$

also has exactly one root point, i.e.,

$$x_1 = \frac{\log\left(\frac{16n}{15}\right) + m\log\left(\frac{16}{15}\right) + \log\log\left(\frac{143}{135}\right) - \log\log\left(\frac{11}{9}\right)}{\log\left(\frac{15}{13}\right)}.$$

So $f(x)$ is strictly decreasing on $(-\infty, x_1]$, and strictly increasing on $[x_1, \infty)$. Moreover, it is easy to see that $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x) = \infty$. For the case $x_1 \le 0$, we will have $f(y_2) < f(y_1)$ for any $0 \le y_2 < y_1$, while for the case $0 < x_1$, there is a point M with coordinates $(x_h, f(x_h))$, at which the function $f(x)$ has the same value as at $x = 0$, i.e., $f(x_h) = f(0)$. Consequently, for any $y_1 \ge x_h$ and $0 \le y_2 < y_1$, we will always have $f(y_1) > f(y_2)$. See [4, 8] with similar details.

In view of Lemma 3.4, the lower bound of $[\mathrm{CD}_2(d(\gamma))]^2$ can be achieved using Lemma 3.4 only if all parts of (3.2) are positive. In order to get the lower bound of $[\mathrm{CD}_2(d(\gamma))]^2$ (Theorem 4.2, given below), we need to introduce the following condition under which we can use Lemma 3.4 to find the lower bound of $[\mathrm{CD}_2(d(\gamma))]^2$.

Lemma 4.1 *For any four-level design* $d \in U(n; 4^m)$, *let* $h = \lfloor \frac{m}{2} \rfloor$ *and*

$$g(r_{11}, r_{22}, \ldots, r_{nn}) = -\frac{2}{n}\left(\frac{135}{128}\right)^m \sum_{i=1}^n \left(\frac{143}{135}\right)^{r_{ii}} + \frac{1}{2n^2}\left(\frac{9}{8}\right)^m \sum_{i=1}^n \left(\frac{11}{9}\right)^{r_{ii}}.$$

Then the lower bound of $g(r_{11}, r_{22}, \ldots, r_{nn})$ *can be achieved using Lemma* 3.4 *only if* $f(h) \ge f(0)$.

Proof The proof is similar to the proof of Lemma 4.1 in [4]. □

Now, we are in a position to get the following lower bound of $[\mathrm{CD}_2(d(\gamma))]^2$.

Theorem 4.2 *For any four-level design* $d \in U(n; 4^m)$ *and for any foldover plan* $\gamma = (\gamma_1, \ldots, \gamma_m) \in \Gamma_t$, *let* $\tau = (\frac{143}{135}), \sigma = (\frac{11}{9}), a = \ln(\frac{11}{8}), b = \ln(\frac{9}{8}), h = \lfloor \frac{m}{2} \rfloor, h_1 = \lfloor \frac{m-t}{2} \rfloor, h_2 = \lfloor \frac{(m-t)+\rho}{2} \rfloor, h_3 = \lfloor \frac{m(n-4)}{8(n-1)} \rfloor, h_4 = \lfloor \frac{m(3n-4)}{8(n-1)} \rfloor, h_5 = \lfloor \frac{m(n-4)+4t}{8(n-1)} \rfloor$ *and* $h_6 = \lfloor \frac{3nm-4(\rho+m-t)}{8(n-1)} \rfloor$. *If*

$$f(h) \ge f(0),$$

then

$$[\mathrm{CD}_2(d(\gamma))]^2 \geq \mathrm{LBC}(t),$$

where

$$\begin{aligned}\mathrm{LBC}(t) = &\left(\frac{13}{12}\right)^m - \frac{2}{n}\left(\frac{135}{128}\right)^m \left[p\tau^h + q\tau^{h+1}\right] + \frac{1}{2n^2}\left(\frac{9}{8}\right)^m \left[p\sigma^h + q\sigma^{h+1}\right] \\ &+\frac{1}{2n^2}\begin{cases} q_1 \mathrm{e}^{w_1} + q_2\mathrm{e}^{w_2} + (p_1 - q_2)\mathrm{e}^{w_3}, \ p_1 > q_2; \\ p_2\mathrm{e}^{w_1} + p_1\mathrm{e}^{w_2} + (q_2 - p_1)\mathrm{e}^{w_4}, \ p_1 \leq q_2; \end{cases} \\ &+\frac{1}{2n^2}\begin{cases} q_3 \mathrm{e}^{v_1} + q_4\mathrm{e}^{v_2} + (p_3 - q_4)\mathrm{e}^{v_3}, \ p_3 > q_4; \\ p_4\mathrm{e}^{v_1} + p_3\mathrm{e}^{v_2} + (q_4 - p_3)\mathrm{e}^{v_4}, \ p_3 \leq q_4; \end{cases} \\ &+\frac{1}{2n^2}\begin{cases} q_5 \mathrm{e}^{u_1} + q_6\mathrm{e}^{u_2} + (p_5 - q_6)\mathrm{e}^{u_3}, \ p_5 > q_6; \\ p_6\mathrm{e}^{u_1} + p_5\mathrm{e}^{u_2} + (q_6 - p_5)\mathrm{e}^{u_4}, \ p_5 \leq q_6, \end{cases}\end{aligned} \tag{4.1}$$

$w_1 = a(h_1+1) + bh_2,\ w_2 = ah_1 + b(h_2+1),\ w_3 = ah_1 + bh_2,\ w_4 = a(h_1+1) + b(h_2+1),$ $v_1 = a(h_3+1) + bh_4,\ v_2 = ah_3 + b(h_4+1),\ v_3 = ah_3 + bh_4,\ v_4 = a(h_3+1) + b(h_4+1), u_1 = a(h_5+1) + bh_6,\ u_2 = ah_5 + b(h_6+1),\ u_3 = ah_5 + bh_6,\ u_4 = a(h_5+1) + b(h_6+1), ph + q(h+1) = \frac{1}{2}mn,\ p_1h_1 + q_1(h_1+1) = \frac{1}{2}n(m-t),\ p_2h_2 + q_2(h_2+1) = \frac{n}{2}(\rho + m - t),\ p_3h_3 + q_3(h_3+1) = \frac{1}{8}mn(n-4),\ p_4h_4 + q_4(h_4+1) = \frac{1}{8}(mn(n-4) + 2mn^2),\ p_5h_5 + q_5(h_5+1) = \frac{1}{8}mn(n-4) + 4tn,\ p_6h_6 + q_6(h_6+1) = \frac{1}{8}\left[3n^2m + 4n(\rho + m - t)\right],\ p + q = p_i + q_i = n,\ i = 1, 2$ *and* $p_j + q_j = n(n-1),\ j = 3, 4, 5, 6.$

Proof From Lemma 3.1, we get

$$\sum_{i=1}^n \left(\frac{143}{135}\right)^{r_{ii}} = \sum_{i=1}^n \mathrm{e}^{cr_{ii}} = \sum_{i=1}^n \sum_{\ell=0}^\infty \frac{(cr_{ii})^\ell}{\ell!},$$

where $c = \log\left(\frac{143}{135}\right)$. Now, from Lemma 3.4, we get

$$\begin{aligned}\sum_{i=1}^n \left(\frac{143}{135}\right)^{r_{ii}} &\geq \sum_{\ell=0}^\infty \frac{p(ch)^\ell + q(c(h+1))^\ell}{\ell!} \\ &= p\mathrm{e}^{ch} + q\mathrm{e}^{c(h+1)} = p\left(\frac{143}{135}\right)^h + q\left(\frac{143}{135}\right)^{h+1}.\end{aligned}$$

Similarly,

$$\sum_{i=1}^n \sum_{j\neq i}^n \left(\frac{11}{8}\right)^{r_{ij}} \left(\frac{9}{8}\right)^{\xi_{ij}} = \sum_{i=1}^n \sum_{j\neq i}^n \mathrm{e}^{ar_{ii} + b\xi_{ij}} = \sum_{i=1}^n \sum_{j\neq i}^n \sum_{\ell=0}^\infty \frac{(ar_{ii} + b\xi_{ij})^\ell}{\ell!}.$$

Now, from Lemma 3.5, we get

$$\begin{aligned}\sum_{i=1}^n \sum_{j\neq i}^n \left(\frac{11}{8}\right)^{r_{ij}} \left(\frac{9}{8}\right)^{\xi_{ij}} &\geq \begin{cases} \displaystyle\sum_{\ell=0}^\infty \frac{q_3{v_1}^\ell + q_4{v_2}^\ell + (p_3 - q_4){v_3}^\ell}{\ell!}, \ p_3 > q_4; \\ \displaystyle\sum_{\ell=0}^\infty \frac{p_4{v_1}^\ell + p_3{v_2}^\ell + (q_4 - p_3){v_4}^\ell}{\ell!}, \ p_3 \leq q_4; \end{cases} \\ &= \begin{cases} q_3\mathrm{e}^{v_1} + q_4\mathrm{e}^{v_2} + (p_3 - q_4)\mathrm{e}^{v_3}, \ p_3 > q_4; \\ p_4\mathrm{e}^{v_1} + p_3\mathrm{e}^{v_2} + (q_4 - p_3)\mathrm{e}^{v_4}, \ p_3 \leq q_4. \end{cases}\end{aligned}$$

Utilizing the same technique as above, we can get the other terms of the right-hand side of (4.1). Hence proved. □

As special cases of Theorem 4.2, when $t = 0$, the following corollary gives the lower bound of the squared centered L_2-discrepancy for four-level U-type designs $d \in U(n; 4^m)$.

Corollary 4.3 *For any four-level design* $d \in U(n; 4^m)$, *let* $\tau = (\frac{143}{135}), \sigma = (\frac{11}{9}), a = \ln(\frac{11}{8}), b = \ln(\frac{9}{8}), h = \lfloor \frac{m}{2} \rfloor$, $h_1 = \lfloor \frac{m(n-4)}{8(n-1)} \rfloor$, $h_2 = \lfloor \frac{m(3n-4)}{8(n-1)} \rfloor$ *and* $g(x) = \frac{2}{9n^2}(\frac{9}{8})^m(\frac{11}{9})^x - \frac{16}{135n}(\frac{135}{128})^m(\frac{143}{135})^x$. *If* $g(h) \geq g(0)$, *then we have*

$$[\mathrm{CD}_2(d)]^2 \geq \mathrm{LBC}(n, m),$$

where

$$\mathrm{LBC}(n,m) = \left(\frac{13}{12}\right)^m - \frac{2}{n}\left(\frac{135}{128}\right)^m \left[p\tau^h + q\tau^{h+1}\right] + \frac{1}{n^2}\left(\frac{9}{8}\right)^m \left[p\sigma^h + q\sigma^{h+1}\right] + \frac{1}{n^2}\begin{cases} q_1 \mathrm{e}^{w_1} + q_2 \mathrm{e}^{w_2} + (p_1 - q_2)\mathrm{e}^{w_3}, & p_1 > q_2; \\ p_2 \mathrm{e}^{w_1} + p_1 \mathrm{e}^{w_2} + (q_2 - p_1)\mathrm{e}^{w_4}, & p_1 \leq q_2, \end{cases}$$

$p + q = n$, $p_i + q_i = n(n-1)$, $i = 1, 2$, $w_1 = a(h_1 + 1) + bh_2$, $w_2 = ah_1 + b(h_2 + 1)$, $w_3 = ah_1 + bh_2$, $w_4 = a(h_1 + 1) + b(h_2 + 1)$, $ph + q(h + 1) = \frac{1}{2}mn$, $p_1 h_1 + q_1(h_1 + 1) = \frac{1}{8}mn(n - 4)$ *and* $p_2 h_2 + q_2(h_2 + 1) = \frac{1}{8}mn(3n - 4)$.

Remark 4.4 According to the property of $f(x)$, the condition $f(h) \geq f(0)$ is trivial when the zero point of $\frac{df(x)}{dx}$, x_1, is less than or equal to zero. When $x_1 > 0$, the condition $f(h) \geq f(0)$ is actually a constraint to m and n. In Table 1, we list the smallest m which satisfies the constraint for different given (small) values of n.

Remark 4.5 The lower bound $\mathrm{LBC}(n, m)$ in Corollary 4.3 is the same as that in [2].

n	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64	68	72
m	6	14	18	20	22	24	24	26	26	28	28	29	30	30	30	31	32	32

Table 1 Smallest m for given n (four-level combined designs)

5 An Algorithm of Searching Optimal Four-level Foldover Plans

In this section, a general algorithm of searching optimal foldover plans for a given design $d \in U(n; 4^m)$ based on Theorems 3.2 and 4.2 can be specified as follows.

Algorithm 1 Searching the optimal foldover four-level plans.

Input the original design $d \in U(n; 4^m)$;
Generate the foldover plan space $\Gamma = \bigcup_{t=0}^{m} \Gamma_t$;
Set the current optimal foldover plan $\gamma_0^* = (0, \ldots, 0)$
Set $\mathrm{CD}_0 = [\mathrm{CD}_2(d(\gamma_0^*))]^2$;
for $t = 1$ to m **do**
for $k = 1$ to $|\Gamma_t|$ **do**
    Compute $[\mathrm{CD}_2(d(\Gamma_t(k)))]^2$, where $\Gamma_t(k)$ is the k-th element of Γ_t;
    if $[\mathrm{CD}_2(d(\Gamma_t(k)))]^2 = \mathrm{LBC}(t)$

Renew $\gamma_0^* = \Gamma_t(k)$;
Renew $\mathrm{CD}_0 = [\mathrm{CD}_2(d(\Gamma_t(k)))]^2$;
break the current **for** loop
elseif $[\mathrm{CD}_2(d(\Gamma_t(k)))]^2 < \mathrm{CD}_0$
Renew $\gamma_0^* = \Gamma_t(k)$;
Renew $\mathrm{CD}_0 = [\mathrm{CD}_2(d(\Gamma_t(k)))]^2$;
end if
end for
end for
Output γ_0^* and $[\mathrm{CD}_2(d(\gamma_0^*))]^2$.

6 Illustrative Examples

In this section depending on Algorithm 1 in Section 5, numerical examples are provided where numerical studies lend further support to our theoretical results. In the following tables (see Tables 2 and 3) one of the optimal foldover plan γ^* over the full foldover plan set Γ of d and the corresponding minimum squared centered L_2-discrepancy value, lower bound are marked by underline and bold.

Example 6.1 Consider the design $d_1 \in U(4; 4^8)$, given below, with $n = 4$ and $m = 8$. See Table 2.

$$d_1 = \begin{bmatrix} 2 & 3 & 2 & 0 & 1 & 2 & 3 & 0 \\ 1 & 0 & 3 & 1 & 2 & 3 & 0 & 2 \\ 0 & 2 & 0 & 2 & 3 & 0 & 1 & 1 \\ 3 & 1 & 1 & 3 & 0 & 1 & 2 & 3 \end{bmatrix}.$$

t	$[\mathrm{CD}_2(d(\gamma^*))]^2$	LBC(t)	γ^*
0	0.5018	0.5018	0 0 0 0 0 0 0 0
1	0.4105	0.3818	2 0 0 0 0 0 0 0
2	0.3334	0.2852	2 2 0 0 0 0 0 0
3	0.2923	0.2159	2 2 2 0 0 0 0 0
4	0.2669	0.1606	2 2 2 0 0 0 0 2
5	0.2613	0.1306	2 2 1 3 0 0 2 0
<u>6</u>	**<u>0.2561</u>**	**<u>0.0443</u>**	**<u>1 2 1 3 1 0 2 0</u>**
7	0.2640	0.0433	1 1 3 1 1 1 3 0
8	0.2772	0.0160	1 3 3 3 1 1 1 1

Table 2 The numerical results of d_1

Example 6.2 Consider the design $d_2 \in U(4; 4^9)$, given below, with $n = 4$ and $m = 9$. See

Table 3.

$$d_2 = \begin{bmatrix} 2 & 3 & 2 & 0 & 1 & 2 & 3 & 0 & 2 \\ 1 & 0 & 3 & 1 & 2 & 3 & 0 & 2 & 1 \\ 0 & 2 & 0 & 2 & 3 & 0 & 1 & 1 & 3 \\ 3 & 1 & 1 & 3 & 0 & 1 & 2 & 3 & 0 \end{bmatrix}.$$

t	$[\mathrm{CD}_2(d(\gamma^*))]^2$	LBC(t)	γ^*
0	0.7260	0.7260	0 0 0 0 0 0 0 0 0
1	0.5940	0.5665	2 0 0 0 0 0 0 0 0
2	0.4966	0.4487	0 0 0 0 0 0 2 0 2
3	0.4300	0.3525	2 0 2 0 0 0 0 2 0
4	0.3915	0.2831	2 2 0 0 0 0 2 0 2
5	0.3725	0.2289	2 0 2 0 0 0 2 2 2
6	0.3706	0.1616	1 2 1 3 1 0 2 0 0
7	**0.3676**	**0.1272**	**1 0 3 2 0 1 1 2 1**
8	0.3762	0.1053	2 1 3 1 1 2 3 0 2
9	0.3918	0.0751	1 1 3 3 1 1 2 1 1

Table 3 The numerical results of d_2

7 Conclusion and Discussion

Foldover of a fractional factorial design is a quick technique to create a design with twice as many runs, which typically releases aliased factors or interactions. The foldover is an useful technique in construction of factorial designs.

Reversing plus and minus signs of one or more factors is the traditional method to foldover two-level fractional factorial designs. This article develops a new mechanism to foldover designs involving factors with multilevel. By exhaustive search we identify the optimal foldover plans. While the method of reversing signs loses its efficacy when factors in the original design have more than two levels, our method here are good for general s-level fractional factorial designs.

The issue of employing the uniformity criterion measured by the centered L_2-discrepancy to assess the optimal foldover plans for four-level designs was studied. For four-level fractional factorial as the original design, general foldover plans and combined designs under general foldover plans are defined, a new analytical formula and a new lower bound of the centered L_2-discrepancy of four-level combined design under a general foldover plan are also obtained. An algorithm for searching the optimal four-level foldover plans is also developed. We also describe a necessary condition for the existence of an optimal four-level foldover plan meeting this lower bound. Illustrative examples are provided, where numerical studies lend further support to our theoretical results.

On the other hand, as Remark 4.5 pointed out, as a special case of Theorem 4.2, when $t = 0$, we have the same lower bound of the centered L_2-discrepancy for four-level U-type design as given in [2]. Hence, our results are also useful complements to the lower bounds of the centered L_2-discrepancy criterion for four-level $U-$type design, which can be served as benchmarks for searching optimal designs in terms of the centered L_2-discrepancy criterion.

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